Central limit theorem for functionals of two independent fractional Brownian motions

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Abstract

We prove a central limit theorem for functionals of two independent d-dimensional fractional Brownian motions with the same Hurst index H in $(\frac{2}{d+1}, \frac{2}{d})$ using the method of moments.

Keywords: fractional Brownian motion, intersection local time, local time, method of moments.

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1 Introduction

Let $\{B_t^H = (B_t^1, \dots, B_t^d), t \geq 0\}$ be a d-dimensional fractional Brownian motion (fBm) with Hurst index H in $(0,1)$. Let $B^{H,1}$ and $B^{H,2}$ be two independent copies of B^H. If $Hd < 2$, then the intersection local time of $B^{H,1}$ and $B^{H,2}$ exists (see [4]) and can be defined as

$$\alpha(t_1, t_2) = \int_0^{t_1} \int_0^{t_2} \delta(B_u^{H,1} - B_v^{H,2})\, du\, dv,$$

where δ is the Dirac delta function. For any t_1 and t_2 in $\mathbb{R}^+$, define

$$X(t_1, t_2) = B_{t_1}^{H,1} - B_{t_2}^{H,2}.$$

We see that $X = \{X(t_1, t_2), t_1, t_2 \in \mathbb{R}^+\}$ is a $(2, d)$-Gaussian random field and satisfies the following scaling property: for any $c > 0$,

$$\{X(ct_1, ct_2), t_1, t_2 \in \mathbb{R}^+\} \overset{\mathcal{L}}{=} \{c^H X(t_1, t_2), t_1, t_2 \in \mathbb{R}^+\}. \tag{1.1}$$

If $Hd < 2$, then, for any x in $\mathbb{R}^d$ and rectangle $E = [a_1, b_1] \times [a_2, b_2]$ in $\mathbb{R}_+^2$, the local time $L(x, E)$ of X exists and is continuous in x, see [6]. When $E = [0, t_1] \times [0, t_2]$, $\alpha(t_1, t_2) = L(0, E)$. Throughout this paper, we assume $Hd < 2$ to ensure the existence and the continuity of $L(x, E)$.

For any integrable function $f : \mathbb{R}^d \to \mathbb{R}$, one can easily show the following convergence in law in the space $C([0, \infty)^2)$, as n tends to infinity,

$$\Big\{n^{Hd-2} \int_0^{nt_1} \int_0^{nt_2} f(B_u^{H,1} - B_v^{H,2})\, du\, dv,\ t_1, t_2 > 0\Big\} \xrightarrow{\mathcal{L}} \Big\{\alpha(t_1, t_2) \int_{\mathbb{R}^d} f(x)\, dx,\ t_1, t_2 > 0\Big\}.$$

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In fact, letting $E=[0,t_1]\times[0,t_2]$, using the scaling property of the process $X(u,v)$ in (1.1) and then applying the continuity of $L(x,E)$, we get

$$\begin{aligned}
n^{Hd-2}\int_0^{nt_1}\int_0^{nt_2} f(B_u^{H,1}-B_v^{H,2})\,du\,dv \quad &\overset{\mathcal{L}}{=} \quad n^{Hd}\int_0^{t_1}\int_0^{t_2} f\big(n^H(B_u^{H,1}-B_v^{H,2})\big)\,du\,dv\\
&= \quad n^{Hd}\int_{\mathbb{R}^d} f(n^H x)L(x,E)\,dx\\
&= \quad \int_{\mathbb{R}^d} f(x)L(\frac{x}{n^H},E)\,dx\\
&\xrightarrow{a.s.} \quad \alpha(t_1,t_2)\int_{\mathbb{R}^d} f(x)\,dx,
\end{aligned}$$

where $\xrightarrow{a.s.}$ denotes the almost sure convergence.

If we assume $\int_{\mathbb{R}^d} f(x)\,dx=0$, then the random variable

$$n^{Hd-2}\int_0^{nt_1}\int_0^{nt_2} f(B_u^{H,1}-B_v^{H,2})\,du\,dv$$

converges in law to 0 as n tends to infinity. It is natural to ask if there is a $\beta>Hd-2$ such that

$$n^{\beta}\int_0^{nt_1}\int_0^{nt_2} f(B_u^{H,1}-B_v^{H,2})\,du\,dv$$

converges to a nontrivial random variable. This will be proved to be true. In order to formulate this result we introduce the following space of functions. Fix a number $\beta\in(0,2)$, define

$$H_0^{\beta}=\Big\{f\in L^1(\mathbb{R}^d):\int_{\mathbb{R}^d}|f(x)||x|^{\beta}\,dx<\infty \quad \text{and} \quad \int_{\mathbb{R}^d} f(x)\,dx=0\Big\}.$$

For any $f\in H_0^{\beta}$, by Lemma 4.1 in [3], the quantity

$$\|f\|_{\beta}^2=-\int_{\mathbb{R}^{2d}} f(x)f(y)|y-x|^{-\beta}\,dx\,dy$$

is finite and nonnegative. The next theorem is the main result of this paper.

Theorem 1.1 *Suppose* $\frac{2}{d+1}<H<\frac{2}{d}$ *and* $f\in H_0^{\frac{2}{H}-d}$. *Then, for any* t_1 *and* $t_2>0$,

$$n^{\frac{Hd-2}{2}}\int_0^{nt_1}\int_0^{nt_2} f(B_u^{H,1}-B_v^{H,2})\,du\,dv \xrightarrow{\mathcal{L}} \sqrt{D_{H,d}}\,\|f\|_{\frac{2}{H}-d}\sqrt{\alpha(t_1,t_2)}\,\zeta,$$

as $n\to\infty$, *where*

$$D_{H,d}=\frac{4}{(2\pi)^{\frac{d}{2}}}\int_0^{\infty}\int_0^{\infty}(u^{2H}+v^{2H})^{-\frac{d}{2}}\big(1-e^{-\frac{1}{2}\frac{1}{u^{2H}+v^{2H}}}\big)du\,dv$$

and ζ *is a standard normal random variable independent of the processes* $B^{H,1}$ *and* $B^{H,2}$.

In [3], Hu, Nualart and I proved the following functional central limit theorem

$$\Big\{n^{\frac{Hd-1}{2}}\int_0^{nt} f(B^H(s))\,ds\,,\ t\geq 0\Big\} \quad \xrightarrow{\mathcal{L}} \quad \Big\{\sqrt{C_{H,d}}\,\|f\|_{\frac{1}{H}-d}\,W(L_t(0))\,,t\geq 0\Big\},$$

where W is a real-valued standard Brownian motion independent of B^H and $L_t(x)$ is the local time of B^H. This paper can be viewed as an extension of the result in [3]. To prove our main result Theorem 1.1, we use the method of moments. Some techniques in [3] will be used, but new ideas are needed. The basic idea of the approach used in this paper is to apply the method of moments to a functional. When dealing with an integral on $[0,t_1]^{2m}\times[0,t_2]^{2m}$, with respect to the measure $du_1\cdots du_{2m}dv_1\cdots dv_{2m}$, we make the change of variables $w_{2k-1}=n(u_{2k}-u_{2k-1})$, $w_{2k}=u_{2k}$, $s_{2k-1}=n(v_{2k}-v_{2k-1})$, $s_{2k}=v_{2k}$, $1\le k\le m$. Then, the increments of $B^{H,1}-B^{H,2}$ in small rectangles will be responsible for the independent noise appearing in the limit. This methodology could be applied to other examples of functionals and multi-parameter processes.

Note that the constant $D_{H,d}$ is finite for any $H>\frac{2}{d+2}$. We conjecture that our result is also true for $\frac{2}{d+2}<H<\frac{2}{d}$, but we have not been able to show our result in the case $H\le\frac{2}{d+2}$. The main reason is that we need to use Fourier analysis in the proof of our result. For example, we need to assume $Hd>2-H$ in Lemma 4.2.

In the Brownian motion case ($H=\frac12$ and $d=3$), the functional version of Theorem 1.1 can be proved using a theorem by Weinryb and Yor [5]. A second order result for two independent Brownian motions in the critical case $d=4$ and $H=\frac12$ was proved by Le Gall [2]. However, not nearly as much has been done for the case $H\neq\frac12$ and $Hd=2$. The general asymptotic results for additive functionals of k independent Brownian motions were obtained by Biane [1]. This paper extends some results in [1] to fractional Brownian motions. General extensions are still largely unknown.

After some preliminaries in Section 2, Section 3 is devoted to the proof of Theorem 1.1, based on the method of moments. Throughout this paper, if not mentioned otherwise, the letter c, with or without a subscript, denotes a generic positive finite constant whose exact value is independent of n and may change from line to line. We use ι to denote $\sqrt{-1}$.

2 Preliminaries

Let $\{B_t^H=(B_t^1,\dots,B_t^d),t\ge0\}$ be a d-dimensional fractional Brownian motion with Hurst index H in $(0,1)$, defined on some probability space $(\Omega,\mathcal{F},P)$. That is, the components of B^H are independent centered Gaussian processes with covariance function

$$\mathbb{E}\left(B_t^iB_s^i\right)=\frac12\big(t^{2H}+s^{2H}-|t-s|^{2H}\big).$$

We shall use the following property of the fractional Brownian motion B^H.

Lemma 2.1 *Given $n\ge1$, there exist two constants c_1 and c_2 depending only on n, H and d, such that for any $0=s_0<s_1<\cdots<s_n$ and $x_i\in\mathbb{R}^d$, $1\le i\le n$, we have*

$$c_1\sum_{i=1}^n|x_i|^2(s_i-s_{i-1})^{2H}\le\mathrm{Var}\Big(\sum_{i=1}^n x_i\cdot(B^H_{s_i}-B^H_{s_{i-1}})\Big)\le c_2\sum_{i=1}^n|x_i|^2(s_i-s_{i-1})^{2H}.$$

Proof. The second inequality is obvious. So it suffices to show the first one, which follows from the local nondeterminism property of the fractional Brownian motion; see, e.g., [1] and [3]. ∎

The inequalities in Lemma 2.1 can be rewritten as

$$c_1\sum_{i=1}^n\Big|\sum_{j=i}^n x_j\Big|^2(s_i-s_{i-1})^{2H}\le\mathrm{Var}\Big(\sum_{i=1}^n x_i\cdot B^H_{s_i}\Big)\le c_2\sum_{i=1}^n\Big|\sum_{j=i}^n x_j\Big|^2(s_i-s_{i-1})^{2H}. \tag{2.1}$$

The next lemma gives a formula for the moments of the random variable $\sqrt{\alpha(t_1,t_2)}\,\zeta$ appearing in Theorem 1.1.

Lemma 2.2 *For any $p \in \mathbb{N}$,*

$$\mathbb{E}\left[\sqrt{\alpha(t_1,t_2)}\,\zeta\right]^p \quad = \quad \begin{cases} \frac{(2m-1)!!}{(2\pi)^{\frac{md}{2}}} \int_{E^m} \left(\det A(u,v)\right)^{-\frac{1}{2}} du\, dv & \text{if } p = 2m, \\ \\ 0 & \text{otherwise,} \end{cases}$$

where $E = [0,t_1] \times [0,t_2]$ and $A(u,v)$ is the covariance matrix of the Gaussian random field

$$\left(B^{H,1}_{u_i} - B^{H,2}_{v_i}, 1 \le i \le m\right).$$

Proof. This follows easily from the properties the normal distribution and the intersection local time $\alpha(t_1,t_2)$. ∎

3 Proof of Theorem 1.1

By the scaling property of $X(t_1,t_2)$ in (1.1), we see that, as random variables,

$$n^{\frac{Hd-2}{2}} \int_0^{nt_1} \int_0^{nt_2} f(B^{H,1}_u - B^{H,2}_v)\, du\, dv \overset{\mathcal{L}}{=} n^{\frac{2+Hd}{2}} \int_0^{t_1} \int_0^{t_2} f\left(n^H (B^{H,1}_u - B^{H,2}_v)\right) du\, dv.$$

Therefore, it suffices to show Theorem 1.1 for the random variable

$$F_n(t_1,t_2) = n^{\frac{2+Hd}{2}} \int_0^{t_1} \int_0^{t_2} f\left(n^H (B^{H,1}_u - B^{H,2}_v)\right) du\, dv.$$

The proof of Theorem 1.1 will be done in two steps. We first show tightness and then establish the convergence of moments.

3.1 Tightness

Tightness will be deduced from the following result.

Proposition 3.1 *For any integer $m \ge 1$, there exists a positive constant C independent of n such that*

$$\mathbb{E}\left[F_n(t_1,t_2)\right]^{2m} \le C \left[\int_{\mathbb{R}^{2d}} |f(x)f(y)||y|^{\frac{2}{H}-d}\, dx\, dy\right]^m.$$

Proof. Note that

$$\mathbb{E}\left[F_n(t_1,t_2)\right]^{2m} = n^{m(2+Hd)} \mathbb{E}\left[\int_{E^{2m}} \prod_{i=1}^{2m} f\left(n^H (B^{H,1}_{u_i} - B^{H,2}_{v_i})\right) du\, dv\right], \tag{3.1}$$

where $E = [0,t_1] \times [0,t_2]$.

Using Fourier analysis and making proper change of variables,

$$
\begin{aligned}
&(2\pi n^H)^{2md}\mathbb{E}\Big[\int_{E^{2m}}\prod_{i=1}^{2m} f\big(n^H(B^{H,1}_{u_i}-B^{H,2}_{v_i})\big)\,du\,dv\Big]\\
&=\int_{\mathbb{R}^{4md}}\int_{E^{2m}}\prod_{i=1}^{2m} f(z_i)\exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{i=1}^{2m}\xi_i\cdot\big(B^{H,1}_{u_i}-B^{H,2}_{v_i}\big)\Big)-\iota\sum_{i=1}^{2m}\frac{z_i\cdot\xi_i}{n^H}\Big\}\,du\,dv\,d\xi\,dz\\
&=\int_{\mathbb{R}^{4md}}\int_{E^{2m}}\prod_{i=1}^{2m} f(z_i)\prod_{i=1}^{2m}\Big(e^{-\iota\frac{z_i\cdot\xi_i}{n^H}}-1\Big)\\
&\qquad\times\exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{i=1}^{2m}\xi_i\cdot B^H_{u_i}\Big)-\frac{1}{2}\mathrm{Var}\Big(\sum_{i=1}^{2m}\xi_i\cdot B^H_{v_i}\Big)\Big\}\,du\,dv\,d\xi\,dz, \qquad (3.2)
\end{aligned}
$$

where in the last equality we used the fact that $\int_{\mathbb{R}^d} f(x)\,dx=0$.

Let $t=\max\{t_1,t_2\}$ and $\mathscr{P}$ be the set consisting of all permutations of $\{1,2,\dots,2m\}$. Set

$$
I_t(\xi)=\int_{[0,t]^{2m}}\exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^{2m}\xi_j\cdot B^H_{u_j}\Big)\Big\}\,du.
$$

For any $\sigma\in\mathscr{P}$, define

$$
I^\sigma_t(\xi)=\int_{D_\sigma}\exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^{2m}\xi_j\cdot B^H_{u_j}\Big)\Big\}\,du,
$$

where $D_\sigma=\{u\in[0,t]^{2m}: u_{\sigma(1)}<\cdots<u_{\sigma(2m)}\}$.

Therefore, $I_t(\xi)$ can be decomposed as

$$
I_t(\xi)=\sum_{\sigma\in\mathscr{P}} I^\sigma_t(\xi). \qquad (3.3)
$$

For simplicity of notation, set

$$
\Phi_n(\xi,z)=n^{m(2-Hd)}\prod_{i=1}^{2m}\big|f(z_i)\big|\prod_{i=1}^{2m}\big|e^{i\frac{z_i\cdot\xi_i}{n^H}}-1\big|. \qquad (3.4)
$$

From (3.1), (3.2) and (3.3), we can write

$$
\mathbb{E}\left[F_n(t_1,t_2)\right]^{2m}\le c_1\int_{\mathbb{R}^{4md}}\Phi_n(\xi,z)\big(I_t(\xi)\big)^2 d\xi\,dz\le c_2\sum_{\sigma\in\mathscr{P}}\int_{\mathbb{R}^{4md}}\Phi_n(\xi,z)\big(I^\sigma_t(\xi)\big)^2\,d\xi\,dz. \qquad (3.5)
$$

Observe that

$$
\begin{aligned}
\big(I^\sigma_t(\xi)\big)^2&=\int_{D_\sigma\times D_\sigma}\exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^{2m}\xi_j\cdot B^H_{u_j}\Big)-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^{2m}\xi_j\cdot B^H_{v_j}\Big)\Big\}\,du\,dv\\
&=\int_{\widehat{D}}\exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^{2m}\xi_j\cdot B^H_{w_j}\Big)-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^{2m}\xi_j\cdot B^H_{s_j}\Big)\Big\}\,dw\,ds,
\end{aligned}
$$

where $\widehat{D}=\big\{w,s\in[0,t]^{2m}: w_1<\cdots<w_{2m}\text{ and } s_1<\cdots<s_{2m}\big\}$.

Using the second inequality in (2.1), we obtain that

$$\int_{\mathbb{R}^{4md}} \Phi_n(\xi, z)\big(I_t^\sigma(\xi)\big)^2 \, d\xi \, dz$$

is less than or equal to

$$\int_{\mathbb{R}^{4md}} \int_{\widehat{D}} \Phi_n(\xi, z) \exp\Big\{ -\frac{\kappa_H}{2} \sum_{i=1}^{2m} |\sum_{j=i}^{2m} \xi_j|^2 \big[(w_i - w_{i-1})^{2H} + (s_i - s_{i-1})^{2H}\big]\Big\} \, dw \, ds \, d\xi \, dz,$$

with the convention $w_0 = s_0 = 0$.

Making the change of variables $\eta_i = \sum\limits_{j=i}^{2m} \xi_j$, $u_i = w_i - w_{i-1}$ and $v_i = s_i - s_{i-1}$ for $i = 1, \dots, 2m$ gives

$$\begin{aligned} &\int_{\mathbb{R}^{4md}} \Phi_n(\xi, z)(I_t^\sigma(\xi))^2 \, d\xi \, dz \\ &\quad \leq n^{m(2-Hd)} \int_{\mathbb{R}^{4md}} \int_{[0,t]^{4m}} |\prod_{i=1}^{2m} f(z_i)| \prod_{i=1}^{2m} \big| \exp\big(\iota \frac{z_i}{n^H} \cdot (\eta_{i+1} - \eta_i)\big) - 1 \big| \\ &\qquad\quad \times \exp\Big\{ -\frac{\kappa_H}{2} \sum_{i=1}^{2m} |\eta_i|^2 (u_i^{2H} + v_i^{2H}) \Big\} \, d\eta \, du \, dv \, dz, \end{aligned} \tag{3.6}$$

with the convention $\eta_{2m+1} = 0$.

Let $\sqrt{\kappa_H} X_1, \dots, \sqrt{\kappa_H} X_{2m}$ be independent copies of the d-dimensional standard normal random vector and $X_{2m+1} = 0$. Then inequality (3.6) can be written as

$$\begin{aligned} &\int_{\mathbb{R}^{4md}} \Phi_n(\xi, z)(I_t^\sigma(\xi))^2 \, d\xi \, dz \\ &\leq c_3 \, n^{m(2-Hd)} \mathbb{E}\Big[\int_{\mathbb{R}^{2md}} \int_{[0,t]^{4m}} \prod_{i=1}^{2m} |f(z_i)| \prod_{i=1}^{2m} (u_i^{2H} + v_i^{2H})^{-\frac{d}{2}} \\ &\qquad \times \prod_{i=1}^{2m} \Big| \exp\Big(\iota \frac{z_i}{n^H} \cdot \big(\frac{X_{i+1}}{\sqrt{u_{i+1}^{2H} + v_{i+1}^{2H}}} - \frac{X_i}{\sqrt{u_i^{2H} + v_i^{2H}}}\big)\Big) - 1 \Big| \, du \, dv \, dz \Big]. \end{aligned} \tag{3.7}$$

To make use of the independence of $X_1, X_2, \dots, X_{2m}$, we replace the terms

$$\Big| \exp\Big(\iota \frac{z_i}{n^H} \cdot \big(\frac{X_{i+1}}{\sqrt{u_{i+1}^{2H} + v_{i+1}^{2H}}} - \frac{X_i}{\sqrt{u_i^{2H} + v_i^{2H}}}\big)\Big) - 1 \Big|, \quad i = 2, 4, \dots, 2m$$

on the right hand side of inequality (3.7) with 2 and then obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^{4md}} \Phi_n(\xi,z)(I_t^\sigma(\xi))^2\, d\xi\, dz \\
&\leq c_4\, n^{m(2-Hd)}\mathbb{E}\Big[\int_{\mathbb{R}^{2md}}\int_{[0,t]^{4m}} \prod_{i=1}^{2m}\big|f(z_i)\big|\prod_{i=1}^{2m}(u_i^{2H}+v_i^{2H})^{-\frac{d}{2}} \\
&\qquad \times \prod_{k=1}^{m}\Big|\exp\Big(\iota\frac{z_i}{n^H}\cdot\big(\frac{X_{2k}}{\sqrt{u_{2k}^{2H}+v_{2k}^{2H}}}-\frac{X_{2k-1}}{\sqrt{u_{2k-1}^{2H}+v_{2k-1}^{2H}}}\big)\Big)-1\Big|\, du\, dv\, dz\Big] \\
&= c_4\Big(n^{2-Hd}\int_{\mathbb{R}^{2d}}\int_{[0,t]^4}\prod_{i=1}^{2}\big|f(z_i)\big|\prod_{i=1}^{2}\big(\sqrt{u_i^{2H}+v_i^{2H}}\big)^{-d} \\
&\qquad \times \mathbb{E}\Big[\Big|\exp\Big(\iota\frac{z_1}{n^H}\cdot\big(\frac{X_2}{\sqrt{u_2^{2H}+v_2^{2H}}}-\frac{X_1}{\sqrt{u_1^{2H}+v_1^{2H}}}\big)\Big)-1\Big|\Big]\, du_1\, du_2\, dv_1\, dv_2\, dz_1\, dz_2\Big)^m \\
&\leq c_5\, t^{2-Hd}\Big(\int_{\mathbb{R}^{2d}}|f(z_1)f(z_2)|z_1|^{\frac{2}{H}-d}\, dz_1\, dz_2\Big)^m, \qquad (3.8)
\end{aligned}
$$

where in the last inequality we used Lemmas 4.1 and 4.2.

Combining (3.8) and (3.5) gives the desired inequality. ∎

3.2 Convergence of odd moments

Let $t=\max\{t_1,t_2\}$. For any $p\in\mathbb{N}$, $y\in\mathbb{R}^d$ and $\xi=(\xi_1,\dots,\xi_p)\in(\mathbb{R}^d)^p$, define

$$
\Phi_{n,p}(\xi,y)=n^{\frac{p(2-Hd)}{2}}\prod_{j=1}^{p}\big|f(y_j)(e^{-\iota\frac{y_j\cdot\xi_j}{n^H}}-1)\big| \qquad (3.9)
$$

and

$$
H_{n,p}=\int_{\mathbb{R}^{2pd}}\int_{[0,t]^{2p}}\Phi_{n,p}(\xi,y)\exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^{p}\xi_j\cdot\big(B_{u_j}^{H,1}-B_{v_j}^{H,2}\big)\Big)\Big\}\, du\, dv\, d\xi\, dy.
$$

Note that for $p=2m$, $\Phi_{n,p}(\xi,y)$ is precisely the function defined (3.4). Also in the proof of Proposition 3.1, we have that $\mathbb{E}\big[F_n(t_1,t_2)\big]^{2m}\leq c\, H_{n,2m}$. We are going to use see that if p is odd, then $H_{n,p}$ converges to zero as n tends to infinity. This will imply the convergence of odd moments.

Proposition 3.2 *If p is odd, then*

$$
\lim_{n\to\infty}H_{n,p}=0.
$$

Proof. Using similar notation as in the proof of Proposition 3.1, we have

$$
\begin{aligned}
H_{n,p}\leq c_1\, n^{\frac{p(2-Hd)}{2}}\,\mathbb{E}\Big[&\int_{\mathbb{R}^{pd}}\int_{[0,t]^{2p}}\prod_{i=1}^{p}\big|f(y_i)\big|\prod_{i=1}^{p}(u_i^{2H}+v_i^{2H})^{-\frac{d}{2}} \\
&\times\prod_{i=1}^{p}\Big|\exp\Big(\iota\frac{y_i}{n^H}\cdot\big(\frac{X_{i+1}}{\sqrt{u_{i+1}^{2H}+v_{i+1}^{2H}}}-\frac{X_i}{\sqrt{u_i^{2H}+v_i^{2H}}}\big)\Big)-1\Big|\, du\, dv\, dy\Big],
\end{aligned}
$$

with the convention $X_{p+1}=0$. Since $X_1, X_2, \cdots, X_p$ are i.i.d,

$$
\begin{aligned}
H_{n,p} \le\; & c_2\, n^{\frac{p(2-Hd)}{2}}\, \mathbb{E}\Big[\int_{\mathbb{R}^{pd}}\int_{[0,t]^{2p}} \prod_{i=1}^p |f(y_i)| \prod_{i=1}^p (u_i^{2H}+v_i^{2H})^{-\frac{d}{2}} \\
& \times \prod_{k=1}^{\frac{p+1}{2}} \Big|\exp\Big(\iota \frac{y_{2k-1}}{n^H}\cdot\big(\frac{X_{2k}}{\sqrt{u_{2k}^{2H}+v_{2k}^{2H}}} - \frac{X_{2k-1}}{\sqrt{u_{2k-1}^{2H}+v_{2k-1}^{2H}}}\big)\Big) - 1\Big|\, ds\, dt\, dy\Big] \\
\le\; & c_3\, n^{\frac{(2-Hd)}{2}} \Big(\int_{\mathbb{R}^{2d}} |f(y_1)f(y_2)| |y_1|^{\frac{2}{H}-d}\, dy_1\, dy_2\Big)^{\frac{p-1}{2}} \\
& \times \mathbb{E}\Big[\int_{\mathbb{R}^d}\int_{[0,t]^2} |f(y_p)| (u_p^{2H}+v_p^{2H})^{-\frac{d}{2}} \Big|\exp\big(-\iota\frac{y_p}{n^H}\cdot \frac{X_p}{\sqrt{u_p^{2H}+v_p^{2H}}}\big) - 1\Big|\, du_p\, dv_p\, dy_p\Big] \\
\le\; & c_4\, n^{\frac{(Hd-2)}{2}} \Big(\int_{\mathbb{R}^{2d}} |f(y_1)f(y_2)| |y_1|^{\frac{2}{H}-d}\, dy_1\, dy_2\Big)^{\frac{p-1}{2}} \Big(\int_{\mathbb{R}^d} |f(y_p)| |y_p|^{\frac{2}{H}-d}\, dy_p\Big),
\end{aligned}
$$

where in the last two inequalities we used Lemmas 4.1 and 4.2. Therefore, $\lim\limits_{n\to\infty} H_{n,p}=0$. ■

Propositions 3.1 and 3.2 show that $H_{n,p}$ is uniformly bounded in n. Moreover, Proposition 3.2 implies the following convergence of odd moments.

Proposition 3.3 *Suppose p is odd, then*

$$\lim_{n\to\infty} \mathbb{E}\,[F_n(t_1,t_2)]^p = 0.$$

Proof. Since $|\mathbb{E}\,(F_n(t_1,t_2))^p| \le H_{n,p}$ for all p, this follows immediately from Proposition 3.2. ■

3.3 Some technical lemmas

To prove the convergence of even moments, we need some technical lemmas.

Recall that

$$\mathbb{E}\big[F_n(t_1,t_2)\big]^{2m} = n^{m(2+Hd)}\mathbb{E}\Big[\int_{E^{2m}} \prod_{j=1}^{2m} f\big(n^H B^{H,1}_{u_j} - n^H B^{H,2}_{v_j}\big)\, du\, dv\Big],$$

where $E=[0,t_1]\times[0,t_2]$.

Let $\mathscr{P}$ be the set consisting of all permutations of $I=\{1,2,\dots,2m\}$ and

$$D = \Big\{0=u_0<u_1<u_2<\cdots<u_{2m}<t_1,\ 0=v_0<v_1<v_2<\cdots<v_{2m}<t_2\Big\}. \tag{3.10}$$

Then

$$\mathbb{E}\big[F_n(t_1,t_2)\big]^{2m} = (2m)!\, n^{m(2+Hd)} \sum_{\sigma\in\mathscr{P}} \mathbb{E}\Big[\int_D \prod_{j=1}^{2m} f\big(n^H B^{H,1}_{u_j} - n^H B^{H,2}_{v_{\sigma(j)}}\big)\, du\, dv\Big]. \tag{3.11}$$

For any $\epsilon>0$, define

$$H^{\sigma}_{n,2m,\epsilon} = \int_{\mathbb{R}^{4md}}\int_{R_\epsilon} \Phi_{n,2m}(\xi,y) \exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^{2m}\xi_j\cdot\big(B^{H,1}_{u_j} - B^{H,2}_{v_{\sigma(j)}}\big)\Big)\Big\}\, du\, dv\, d\xi\, dy,$$

where

$$R_\epsilon = \Big\{0 < u_1 < u_2 < \cdots < u_{2m} < t,\ 0 < v_1 < v_2 < \cdots < v_{2m} < t\Big\}$$
$$\bigcap \Big\{|u_{2j} - u_{j-2}| < \epsilon \text{ or } |v_{2j} - v_{2j-2}| < \epsilon, \text{ for some } j \in \{1, 2, \dots, m\}\Big\}$$

with the convention $u_0 = v_0 = 0$ and $t = \max\{t_1, t_2\}$.

Lemma 3.4 *For any $\sigma \in \mathscr{P}$,*

$$\lim_{\epsilon\to 0} \sup_n H^\sigma_{n,p,\epsilon} = 0.$$

Proof. Note that

$$R_\epsilon = \cup_{\ell=1}^m \left(R_{\epsilon,\ell,1} \cup R_{\epsilon,\ell,2}\right),$$

where $R_{\epsilon,\ell,1} = R_\epsilon \cap \big\{|u_{2\ell} - u_{2\ell-2}| < \epsilon\big\}$ and $R_{\epsilon,\ell,2} = R_\epsilon \cap \big\{|v_{2\ell} - v_{2\ell-2}| < \epsilon\big\}$. So it suffices to show that

$$\lim_{\epsilon\to 0} \sup_n \int_{\mathbb{R}^{4md}} \int_{R_{\epsilon,\ell,i}} \Phi_{n,2m}(\xi, y) \exp\Big\{ -\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^{2m} \xi_j \cdot \big(B^{H,1}_{u_j} - B^{H,2}_{v_{\sigma(j)}}\big)\Big)\Big\}\, du\, dv\, d\xi\, dy = 0$$

for all $\ell = 1, 2, \dots, m$ and for $i = 1, 2$. We will consider only the case $i = 2$ and the case $i = 1$ could be treated in the same way.

Let

$$J_t(\xi) = \int_{\{0<u_1<u_2<\cdots<u_{2m}<t\}} \exp\Big\{ -\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^{2m} \xi_j \cdot B^H_{u_j}\Big)\Big\}\, du$$

and

$$J^{\sigma,\epsilon}_t(\xi) = \int_{\{0<u_1<u_2<\cdots<u_{2m}<t,\ |v_{2\ell}-v_{2\ell-2}|<\epsilon\}} \exp\Big\{ -\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^{2m} \xi_j \cdot B^H_{v_{\sigma(j)}}\Big)\Big\}\, dv.$$

Applying Cauchy-Schwartz inequality, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{4md}} \int_{R_{\epsilon,\ell,2}} \Phi_{n,2m}(\xi, y) \exp\Big\{ -\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^{2m} \xi_j \cdot \big(B^{H,1}_{u_j} - B^{H,2}_{v_{\sigma(j)}}\big)\Big)\Big\}\, du\, dv\, d\xi\, dy \\
&\le \Big(\int_{\mathbb{R}^{4md}} \Phi_{n,2m}(\xi, y)(J_t(\xi))^2\, d\xi\, dy\Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^{4md}} \Phi_{n,2m}(\xi, y)(J^{\sigma,\epsilon}_t(\xi))^2\, d\xi\, dy\Big)^{\frac{1}{2}} \\
&\le (H_{n,2m})^{\frac{1}{2}} \Big(\int_{\mathbb{R}^{4md}} \Phi_{n,2m}(\xi, y)(J^{\sigma,\epsilon}_t(\xi))^2\, d\xi\, dy\Big)^{\frac{1}{2}}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\int_{\mathbb{R}^{4md}} \Phi_{n,2m}(\xi, y)(J^{\sigma,\epsilon}_t(\xi))^2\, d\xi\, dy \\
&= \int_{\mathbb{R}^{4md}} \int_{R_{\epsilon,\ell}} \Phi_{n,2m}(\xi, y) \exp\Big\{ -\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^{2m} \xi_j \cdot \big(B^{H,1}_{u_j} - B^{H,2}_{v_j}\big)\Big)\Big\}\, du\, dv\, d\xi\, dy,
\end{aligned}$$

where

$$R_{\epsilon,\ell} = R_{\epsilon,\ell,1} \cap R_{\epsilon,\ell,2}.$$

Since $H_{n,2m}$ is uniformly bounded in n, we only need to show that

$$\lim_{n\to\infty}\int_{\mathbb{R}^{4md}}\int_{R_{\epsilon,\ell}}\Phi_{n,2m}(\xi,y)\exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^{2m}\xi_j\cdot\big(B^{H,1}_{u_j}-B^{H,2}_{v_j}\big)\Big)\Big\}\,du\,dv\,d\xi\,dy=0$$

for all $\ell=1,2,\dots,m$.

Using Lemma 2.1 and then making the change of variables $w_i=u_i-u_{i-1}$, $s_i=v_i-v_{i-1}$ and $\eta_i=\sum_{j=i}^{2m}\xi_j$ for $i=1,2,\dots,2m$ with the convention $u_0=v_0=0$ and $\eta_{2m+1}=0$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{4md}}\int_{R_{\epsilon,\ell}}\Phi_{n,2m}(\xi,y)\exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^{2m}\xi_j\cdot\big(B^{H,1}_{u_j}-B^{H,2}_{v_j}\big)\Big)\Big\}\,du\,dv\,d\xi\,dy\\
&\le n^{m(2-Hd)}\int_{\mathbb{R}^{4md}}\int_{\widehat{R}_{\epsilon,\ell}}\prod_{i=1}^{2m}|f(y_i)|\prod_{i=1}^{2m}\big|\exp\big(\iota\frac{y_i}{n^H}\cdot(\eta_{i+1}-\eta_i)\big)-1\big|\\
&\qquad\times\exp\Big\{-\frac{\kappa_H}{2}\sum_{i=1}^{2m}|\eta_i|^2(w_i^{2H}+s_i^{2H})\Big\}\,dw\,ds\,d\eta\,dy,
\end{aligned}\tag{3.12}$$

where

$$\widehat{R}_{\epsilon,\ell}=[0,t]^{4m}\cap\Big\{\sum_{i=1}^{2m}w_i<t,\ \sum_{i=1}^{2m}s_i<t,\ w_{2\ell}+w_{2\ell-1}<\epsilon,\ s_{2\ell}+s_{2\ell-1}<\epsilon\Big\}.$$

Using the same argument as in the proof of Proposition 3.1, we can prove that the right hand side of the inequality (3.12) is less than a constant multiple of ϵ^{2-Hd}. Letting $\epsilon\to0$ completes the proof. ∎

For $1\le k\le m$, we define

$$\begin{aligned}
O_{2m,k}=\Big\{&u,v\in[0,t]^{2m}:u_1<u_2<\cdots<u_{2m},\ v_1<v_2<\cdots<v_{2m},\\
&\frac{u_{2k}-u_{2k-2}}{2}<u_{2k}-u_{2k-1}\ \text{or}\ \frac{v_{2k}-v_{2k-2}}{2}<v_{2k}-v_{2k-1}\Big\}.
\end{aligned}$$

Recall the definition of $\Phi_{n,2m}(\xi,y)$ in (3.9). The following result states that the integral over the domain $O_{2m,k}$ does not contribute to the limit of the $2m$-th moment, which will play a fundamental role in computing the limits of even moments.

Lemma 3.5 *For any $1\le k\le m$,*

$$\lim_{n\to\infty}\int_{\mathbb{R}^{4md}}\int_{O_{2m,k}}\Phi_{n,2m}(\xi,y)\exp\Big\{-\frac{1}{2}\mathrm{Var}\big(\sum_{j=1}^{2m}\xi_j\cdot(B^{H,1}_{u_j}-B^{H,2}_{v_j})\big)\Big\}\,d\xi\,du\,dv\,dy=0.$$

Proof. Define

$$\begin{aligned}
\widehat{O}_{2m,k}=\big\{&u,v\in[0,t]^{2m}:u_1<u_2<\cdots<u_{2m},v_1<v_2<\cdots<v_{2m},\\
&\frac{u_{2k}-u_{2k-2}}{2}<u_{2k}-u_{2k-1},\frac{v_{2k}-v_{2k-2}}{2}<v_{2k}-v_{2k-1}\big\}.
\end{aligned}$$

Using Cauchy-Schwartz inequality, we obtain

$$\int_{\mathbb{R}^{4md}}\int_{O_{2m,k}} \Phi_{n,2m}(\xi,y)\exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^{2m}\xi_j\cdot(B^{H,1}_{u_j}-B^{H,2}_{v_j})\Big)\Big\}\,d\xi\,du\,dv\,dy$$
$$\le (H_{n,2m})^{\frac{1}{2}}\Big(\int_{\mathbb{R}^{4md}}\Phi_{n,2m}(\xi,y)\int_{\widehat{O}_{2m,k}}\exp\Big\{-\frac{1}{2}\mathrm{Var}\big(\sum_{j=1}^{2m}\xi_j\cdot(B^{H,1}_{u_j}-B^{H,2}_{v_j})\big)\Big\}\,du\,dv\,d\xi\,dy\Big)^{\frac{1}{2}}$$
$$\le c_1\Big(\int_{\mathbb{R}^{4md}}\Phi_{n,2m}(\xi,y)\int_{\widehat{O}_{2m,k}}\exp\Big\{-\frac{1}{2}\mathrm{Var}\big(\sum_{j=1}^{2m}\xi_j\cdot(B^{H,1}_{u_j}-B^{H,2}_{v_j})\big)\Big\}\,du\,dv\,d\xi\,dy\Big)^{\frac{1}{2}}.$$

So it suffices to show

$$\lim_{n\to\infty}\int_{\mathbb{R}^{4md}}\Phi_{n,2m}(\xi,y)\int_{\widehat{O}_{2m,k}}\exp\Big\{-\frac{1}{2}\mathrm{Var}\big(\sum_{j=1}^{2m}\xi_j\cdot(B^{H,1}_{u_j}-B^{H,2}_{v_j})\big)\Big\}\,du\,dv\,d\eta\,dy=0.$$

For $j=1,2,\dots,2m$, we make the change of variables $w_j=u_j-u_{j-1}$ and $s_j=v_j-v_{j-1}$ with the convention $u_0=v_0=0$. For $k=1,2,\dots,m$, define

$$D_{2m,k}=\Big\{w,s\in[0,t]^{2m}:\sum_{j=1}^{2m}w_j<t,\ \sum_{j=1}^{2m}s_j<t,\ w_{2k-1}<w_{2k},\ s_{2k-1}<s_{2k}\Big\}.$$

Using the second inequality in (2.1),

$$\int_{\mathbb{R}^{4md}}\int_{\widehat{O}_{2m,k}}\Phi_{n,2m}(\xi,y)\exp\Big\{-\frac{1}{2}\mathrm{Var}\big(\sum_{j=1}^{2m}\xi_j\cdot B^H_{u_j}\big)-\frac{1}{2}\mathrm{Var}\big(\sum_{j=1}^{2m}\xi_j\cdot B^H_{v_j}\big)\Big\}\,du\,dv\,d\xi\,dy$$
$$\le c_2\,n^{m(2-Hd)}\int_{\mathbb{R}^{2md}}\int_{D_{2m,k}}\prod_{j=1}^{2m}|f(y_j)|\prod_{j=1}^{2m}\big(w_j^{2H}+s_j^{2H}\big)^{-\frac{d}{2}}$$
$$\times\,\mathbb{E}\Big(\prod_{j=1}^{2m}\Big|\exp\Big(\iota\frac{y_j\cdot X_{j+1}}{n^H\sqrt{w_{j+1}^{2H}+s_{j+1}^{2H}}}-\iota\frac{y_j\cdot X_j}{n^H\sqrt{w_j^{2H}+s_j^{2H}}}\Big)-1\Big|\Big)\,dw\,ds\,dy,$$

where $\sqrt{\kappa_H}X_j$ $(1\le j\le 2m)$ are independent copies of the d-dimensional standard normal random vector and $X_{2m+1}=0$. The rest of proof is similar to that of Proposition 3.3 in [3]. ∎

Recall the definition of D in (3.10). For $\ell=1,2,\dots,m$ and $K>0$, define

$$D^n_{K,\ell}=D\cap\big\{u_{2\ell}-u_{2\ell-1}\ge K/n\ \text{ or }\ v_{2\ell}-v_{2\ell-1}\ge K/n\big\}. \tag{3.13}$$

The following result implies that the domain $D^n_{K,\ell}$ does not contribute to the limit of even moments.

Lemma 3.6 *For any $\sigma\in\mathscr{P}$ and $\ell=1,2,\dots,m$,*

$$\lim_{K\to\infty}\limsup_{n\to\infty}\int_{\mathbb{R}^{4md}}\int_{D^n_{K,\ell}}\Phi_{n,2m}(\xi,y)\exp\Big\{-\frac{1}{2}\mathrm{Var}\Big(\prod_{j=1}^{2m}\xi_j\cdot(B^{H,1}_{u_j}-B^{H,2}_{v_{\sigma(j)}})\Big)\Big\}\,du\,dv\,d\xi dy=0.$$

Proof. Let $t = \max(t_1, t_2)$. Define

$$\widehat{D}^n_{K,\ell} = \big\{u, v \in [0,t]^{2m} : u_1 < u_2 < \cdots < u_{2m},\ v_1 < v_2 < \cdots < v_{2m},\\ u_{2\ell} - u_{2\ell-1} \geq K/n,\ v_{2\ell} - v_{2\ell-1} \geq K/n\big\}$$

and

$$I^n_{K,\ell} = \int_{\mathbb{R}^{4md}} \int_{D^n_{K,\ell}} \Phi_{n,2m}(\xi, y) \exp\Big\{ -\frac{1}{2}\mathrm{Var}\Big(\prod_{j=1}^{2m} \xi_j \cdot (B^{H,1}_{u_j} - B^{H,2}_{v_{\sigma(j)}})\Big)\Big\}\, du\, dv\, d\xi\, dy.$$

Applying Cauchy-Schwartz ineuqality,

$$I^n_{K,\ell} \leq c_1 \Big(\int_{\mathbb{R}^{4md}} \int_{\widehat{D}^n_{K,\ell}} \Phi_{n,2m}(\xi, y) \exp\Big\{ -\frac{1}{2}\mathrm{Var}\Big(\sum_{j=1}^{2m} \xi_j \cdot (B^{H,1}_{u_j} - B^{H,2}_{v_j})\Big)\Big\}\, du\, dv\, d\xi\, dy \Big)^{\frac{1}{2}}.$$

According to Lemma 3.5, we can replace $\widehat{D}^n_{K,\ell}$ in the above inequality with

$$\widetilde{D}^n_{K,\ell} = \widehat{D}^n_{K,\ell} \cap \Big\{ \frac{u_{2\ell} - u_{2\ell-2}}{2} > u_{2\ell} - u_{2\ell-1} \geq K/n,\ \frac{v_{2\ell} - v_{2\ell-2}}{2} > v_{2\ell} - v_{2\ell-1} \geq K/n \Big\}.$$

The rest of the proof is similar to that of Proposition 3.4 in [3]. ∎

We next divide $\mathscr{P}$ into two subsets. In section 3.4, we will show that permutations in one subset do not contribute to the convergence of even moments.

For each σ in $\mathscr{P}$, we introduce the following decomposition of $I = \{1, 2, \ldots, 2m\}$:

$$I^\sigma_{ee} = \{j \in I :\ j \text{ is even and } \sigma(j) \text{ is even}\},\quad I^\sigma_{eo} = \{j \in I :\ j \text{ is even and } \sigma(j) \text{ is odd}\},$$
$$I^\sigma_{oe} = \{j \in I :\ j \text{ is odd and } \sigma(j) \text{ is even}\},\quad I^\sigma_{oo} = \{j \in I :\ j \text{ is odd and } \sigma(j) \text{ is odd}\}.$$

Let $I_e = \{2j : 1 \leq j \leq m\}$ and $I_o = \{2j-1 : 1 \leq j \leq m\}$. Then $I_e = I^\sigma_{ee} + I^\sigma_{eo}$ and $I_o = I^\sigma_{oe} + I^\sigma_{oo}$ for all σ in $\mathscr{P}$. We make the change of variables $w_{2k} = u_{2k}$, $w_{2k-1} = n(u_{2k} - u_{2k-1})$, $s_{2k} = v_{2k}$, $s_{2k-1} = n(v_{2k} - v_{2k-1})$ for $k = 1, 2, \ldots, m$. Define

$$D_n = \Big\{ w, s \in \mathbb{R}^{2m} : 0 < w_2 < w_4 < \cdots < w_{2m} < t_1, 0 < s_2 < s_4 < \cdots < s_{2m} < t_2 \\ 0 < w_{2k-1} < n(w_{2k} - w_{2k-1}),\ 0 < s_{2k-1} < n(s_{2k} - s_{2k-1}),\ 1 \leq k \leq m \Big\}. \tag{3.14}$$

From the above decomposition of I,

$$\begin{aligned}
&n^{2m}\mathbb{E}\Big[\int_D \prod_{j=1}^{2m} f\big(n^H B^{H,1}_{u_j} - n^H B^{H,2}_{v_{\sigma(j)}}\big)\, du\, dv \Big] \\
=&\mathbb{E}\Big\{ \int_{D_n} \prod_{j \in I^\sigma_{ee}} f\big(n^H [B^{H,1}_{w_j} - B^{H,2}_{s_{\sigma(j)}}]\big) \prod_{j \in I^\sigma_{eo}} f\big(n^H [B^{H,1}_{w_j} - B^{H,2}_{s_{\sigma(j)+1}}] - n^H [B^{H,2}_{s_{\sigma(j)+1} - \frac{s_{\sigma(j)}}{n}} - B^{H,2}_{s_{\sigma(j)+1}}]\big) \\
&\times \prod_{j \in I^\sigma_{oe}} f\big(n^H [B^{H,1}_{w_{j+1}} - B^{H,2}_{s_{\sigma(j)}}] + n^H [B^{H,1}_{w_{j+1} - \frac{w_j}{n}} - B^{H,1}_{w_{j+1}}]\big) \\
&\quad \times \prod_{j \in I^\sigma_{oo}} f\Big(n^H [B^{H,1}_{w_{j+1}} - B^{H,2}_{s_{\sigma(j)+1}}] + n^H [B^{H,1}_{w_{j+1} - \frac{w_j}{n}} - B^{H,1}_{w_{j+1}}] - n^H [B^{H,2}_{s_{\sigma(j)+1} - \frac{s_{\sigma(j)}}{n}} - B^{H,2}_{s_{\sigma(j)+1}}]\Big)\, dw\, ds \Big\}.
\end{aligned} \tag{3.15}$$

Assume that $x_2, x_4, \dots, x_{2m}$ and $z_2, z_4, \dots, z_{2m}$ are linearly independent elements in some linear space. For any σ in $\mathscr{P}$, let

$$A_{ee}^\sigma = \{x_j - z_{\sigma(j)} : j \in I_{ee}^\sigma\}, \quad A_{oe}^\sigma = \{x_{j+1} - z_{\sigma(j)} : j \in I_{oe}^\sigma\};$$

$$A_{eo}^\sigma = \{x_j - z_{\sigma(j)+1} : j \in I_{eo}^\sigma\}, \quad A_{oo}^\sigma = \{x_{j+1} - z_{\sigma(j)+1} : j \in I_{oo}^\sigma\}.$$

Note that elements in each of the above sets are linearly independent. For simplicity, we use $\#A$ to denote the cardinality of a set A. Suppose $\#I_{ee}^\sigma = r$. Then $\#A_{ee}^\sigma = \#A_{oo}^\sigma = r$ and $\#A_{eo}^\sigma = \#A_{oe}^\sigma = m - r$. We are interested in the dimension of the set $A_\sigma := A_{ee}^\sigma \cup A_{oe}^\sigma \cup A_{eo}^\sigma \cup A_{oo}^\sigma$, that is, the maximum number of elements in A_σ which are linearly independent. Since elements in $A_{ee}^\sigma \cup A_{oe}^\sigma$ are linearly independent, the dimension of A_σ is greater than or equal to m.

Lemma 3.7 *The dimension of A_σ is m if and only if $\{(j,\sigma(j)) : j \in I_{ee}^\sigma\} = \{(j+1, \sigma(j)+1) : j \in I_{oo}^\sigma\}$ and $\{(j+1,\sigma(j)) : j \in I_{oe}^\sigma\} = \{(j, \sigma(j)+1) : j \in I_{eo}^\sigma\}$.*

Proof. It suffices to show the only if part. Note that the m elements in $A_{ee}^\sigma \cup A_{oe}^\sigma$ are linearly independent. If one of the two condition fails, then there must exist an element in $A_{eo}^\sigma \cup A_{oo}^\sigma$ such that it does not belong to the space spanned by $A_{ee}^\sigma \cup A_{oe}^\sigma$. This implies that the dimension of A_σ is greater than m. ∎

Let $\mathscr{P}_0 = \{\sigma \in \mathscr{P} : \text{ the dimension of } A_\sigma \text{ is } m\}$ and $\mathscr{P}_1 = \mathscr{P} - \mathscr{P}_0$. Lemma 3.7 implies

$$\#\mathscr{P}_0 = \sum_{r=0}^m \binom{m}{r} m! = m!\, 2^m.$$

3.4 Convergence of even moments

We will show the convergence of all even moments. Recall that

$$\mathbb{E}\left[F_n(t_1,t_2)\right]^{2m} = (2m)!\, n^{m(2+Hd)} \sum_{\sigma \in \mathscr{P}} \mathbb{E}\Big[\int_D \prod_{j=1}^{2m} f\big(n^H B_{u_j}^{H,1} - n^H B_{v_{\sigma(j)}}^{H,2}\big)\, du\, dv\Big],$$

where

$$D = \Big\{0 = u_0 < u_1 < u_2 < \cdots < u_{2m} < t_1,\, 0 = v_0 < v_1 < v_2 < \cdots < v_{2m} < t_2\Big\}.$$

Note that we can find a sequence of functions f_N, which are infinitely differentiable with compact support, such that $\int_{\mathbb{R}^d} f_N(x)\, dx = 0$ and

$$\lim_{N\to\infty} \int_{\mathbb{R}^d} |f(x) - f_N(x)| \left(|x|^{\frac{2}{H}-d} \vee 1\right) dx = 0.$$

Therefore, by Proposition 3.1, we can assume that f is infinitely differentiable with compact support and $\int_{\mathbb{R}^d} f(x)\, dx = 0$.

We first show that permutations in $\mathcal{P}_1$ do not contribute to the limit of even moments using Lemmas 3.4 and 3.6.

Proposition 3.8 *For any $\sigma \in \mathscr{P}_1$,*

$$\lim_{n\to\infty} n^{m(2+Hd)} \mathbb{E}\Big[\int_D \prod_{j=1}^{2m} f\big(n^H B_{u_j}^{H,1} - n^H B_{v_{\sigma(j)}}^{H,2}\big)\, du\, dv\Big] = 0. \tag{3.16}$$

Proof. For any $\epsilon > 0$ and $K > 0$, define

$D^n_{K,\epsilon} = D \cap \big\{u_{2\ell} - u_{2\ell-1} < K/n,\ v_{2\ell} - v_{2\ell-1} < K/n,\ u_{2\ell} - u_{2\ell-2} \geq \epsilon,\ v_{2\ell} - v_{2\ell-2} \geq \epsilon,\ \ell = 1, 2, \ldots, m\big\}$.

Thanks to Lemmas 3.4 and 3.6, we can replace D in (3.16) with $D^n_{K,\epsilon}$.

Recall the equality in (3.15). Making proper change of variables gives

$$
\begin{aligned}
&n^{m(2+Hd)}\mathbb{E}\Big[\int_{D^n_{K,\epsilon}} \prod_{j=1}^{2m} f\big(n^H B^{H,1}_{u_j} - n^H B^{H,2}_{v_{\sigma(j)}}\big)\, du\, dv\Big] \\
=&\mathbb{E}\Big\{\int_{\widehat{D}^n_{K,\epsilon}} \prod_{j\in I^\sigma_{ee}} f\big(n^H[B^{H,1}_{w_j} - B^{H,2}_{s_{\sigma(j)}}]\big) \prod_{j\in I^\sigma_{eo}} f\big(n^H[B^{H,1}_{w_j} - B^{H,2}_{s_{\sigma(j)+1}}] - n^H[B^{H,2}_{s_{\sigma(j)+1}-\frac{s_{\sigma(j)}}{n}} - B^{H,2}_{s_{\sigma(j)+1}}]\big) \\
&\quad\times \prod_{j\in I^\sigma_{oe}} f\big(n^H[B^{H,1}_{w_{j+1}} - B^{H,2}_{s_{\sigma(j)}}] + n^H[B^{H,1}_{w_{j+1}-\frac{w_j}{n}} - B^{H,1}_{w_{j+1}}]\big) \\
&\qquad\times \prod_{j\in I^\sigma_{oo}} f\Big(n^H[B^{H,1}_{w_{j+1}} - B^{H,2}_{s_{\sigma(j)+1}}] + n^H[B^{H,1}_{w_{j+1}-\frac{w_j}{n}} - B^{H,1}_{w_{j+1}}] - n^H[B^{H,2}_{s_{\sigma(j)+1}-\frac{s_{\sigma(j)}}{n}} - B^{H,2}_{s_{\sigma(j)+1}}]\Big)\, dw\, ds\Big\},
\end{aligned}
$$

where

$$
\begin{aligned}
\widehat{D}^n_{K,\epsilon} = \Big\{&w, s \in \mathbb{R}^{2m} : \epsilon < w_2 < w_4 < \cdots < w_{2m} < t_1,\ \epsilon < s_2 < s_4 < \cdots < s_{2m} < t_2, \\
&0 < w_{2k-1} < K \wedge n(w_{2k} - w_{2k-2}),\ 0 < s_{2k-1} < K \wedge n(s_{2k} - s_{2k-2}), \\
&w_{2k} - w_{2k-2} \geq \epsilon,\ s_{2k} - s_{2k-2} \geq \epsilon,\ k = 2, \ldots, m\Big\}. \qquad (3.17)
\end{aligned}
$$

Since $\sigma \in \mathcal{P}_1$, there exists a $j_0 \in I^\sigma_{eo} \cup I^\sigma_{oo}$ such that $j_0 \notin I^\sigma_{ee} \cup I^\sigma_{oe}$. Without loss of generality, we can assume that $j_0 \in I^\sigma_{eo}$. Let $p_n(x, y)$ be the density function of the random field $Z_n = (X_j, Y^n_j)$ where

$$
X_j = \begin{cases} B^{H,1}_{w_{j_0}} - B^{H,2}_{s_{\sigma(j_0)+1}} & \text{if } j = j_0 \\ B^{H,1}_{w_j} - B^{H,2}_{s_{\sigma(j)}} & \text{if } j \in I^\sigma_{ee} \\ B^{H,1}_{w_{j+1}} - B^{H,2}_{s_{\sigma(j)}} & \text{if } j \in I^\sigma_{oe} \end{cases} \qquad (3.18)
$$

and

$$
Y^n_j = \begin{cases} n^H[B^{H,2}_{s_{\sigma(j_0)+1}-\frac{s_{\sigma(j_0)}}{n}} - B^{H,2}_{s_{\sigma(j_0)+1}}] & \text{if } j = j_0 \\ n^H[B^{H,1}_{w_{j+1}-\frac{w_j}{n}} - B^{H,1}_{w_{j+1}}] & \text{if } j \in I^\sigma_{oe}. \end{cases}
$$

Since f is infinitely differentiable and had compact support,

$$
\begin{aligned}
&n^{m(2+Hd)}\Big|\mathbb{E}\Big[\int_{D^n_{K,\epsilon}} \prod_{j=1}^{2m} f\big(n^H B^{H,1}_{u_j} - n^H B^{H,2}_{v_{\sigma(j)}}\big)\, du\, dv\Big]\Big| \\
&\leq c_1\, n^{mHd}\mathbb{E}\Big[\int_{\widehat{D}^n_{K,\epsilon}} \Big|f\Big(n^H[B^{H,1}_{w_{j_0}} - B^{H,2}_{s_{\sigma(j_0)+1}}] - n^H[B^{H,2}_{s_{\sigma(j_0)+1}-\frac{s_{\sigma(j_0)}}{n}} - B^{H,2}_{s_{\sigma(j_0)+1}}]\Big)\Big| \\
&\quad\times \prod_{j\in I^\sigma_{ee}} \Big|f\Big(n^H[B^{H,1}_{w_j} - B^{H,2}_{s_{\sigma(j)}}]\Big)\Big| \prod_{j\in I^\sigma_{oe}} \Big|f\Big(n^H[B^{H,1}_{w_{j+1}} - B^2_{s_{\sigma(j)}}] + n^H[B^{H,1}_{w_{j+1}-\frac{w_j}{n}} - B^{H,1}_{w_{j+1}}]\Big)\Big|\, dw\, ds\Big] \\
&= c_2\, n^{mHd}\int_{\widehat{D}^n_{K,\epsilon}} \int_{\mathbb{R}^{(m+2+\#I^\sigma_{oe})d}} \big|f(n^H x_{j_0} + y_{j_0})\big| \prod_{j\in I^\sigma_{ee}} \big|f(n^H x_j)\big| \prod_{j\in I^\sigma_{oe}} \big|f(n^H x_j + y_j)\big|\, p_n(x, y)\, dx\, dy\, dw\, ds \\
&= c_2\, n^{-Hd}\int_{\widehat{D}^n_{K,\epsilon}} \int_{\mathbb{R}^{(m+2+\#I^\sigma_{oe})d}} \big|f(x_{j_0} + y_{j_0})\big| \prod_{j\in I^\sigma_{ee}} \big|f(x_j)\big| \prod_{j\in I^\sigma_{oe}} \big|f(x_j + y_j)\big|\, p_n(\frac{x}{n^H}, y)\, dx\, dy\, dw\, ds \\
&\leq c_3\, n^{-Hd}\int_{\widehat{D}^n_{K,\epsilon}} \int_{\mathbb{R}^{(1+\#I^\sigma_{oe})d}} \sup_x p_n(\frac{x}{n^H}, y)\, dy\, dw\, ds.
\end{aligned}
$$

Let $Q_n(w,s)$ be the covariance matrix function of $Z_n = (X_j, Y_j^n)$ defined above. Then Q_n has the following expression

$$Q_n = \left[\begin{array}{cc} A & C_n^T \\ C_n & B_n \end{array} \right],$$

where $A = A(w,s)$ is the covariance matrix function of $X = (X_j)$, $C_n = C_n(w,s)$ the covariance matrix function of $X = (X_j)$ and $Y = (Y_j)$, and $B_n = B_n(w,s)$ the covariance matrix function of $Y = (Y_j)$.

After doing some algebra, we have

$$Q_n^{-1} = \left[\begin{array}{cc} (A - C_n^T B_n^{-1} C_n)^{-1} & -A^{-1} C_n^T (B_n - C_n A^{-1} C_n^T)^{-1} \\ -B_n^{-1} C_n (A - C_n^T B_n^{-1} C_n)^{-1} & (B_n - C_n A^{-1} C_n^T)^{-1} \end{array} \right],$$

and $\det(Q_n) = \det(A)\det(B_n - C_n A^{-1} C_n^T)$. For simplicity of notation, we write

$$Q_n^{-1} = \left[\begin{array}{cc} D_1 & D_2^T \\ D_2 & D_4 \end{array} \right].$$

Note that

$$\begin{aligned}
(x,y) Q_n^{-1} (x,y)^T &= x D_1 x^T + x D_2^T y^T + y D_2 x^T + y D_4 y^T \\
&= x D_1 x^T + 2x D_2^T y^T + y D_4 y^T \\
&= (x\sqrt{D_1})(x\sqrt{D_1})^T + 2(x\sqrt{D_1})(\sqrt{D_1})^{-1} D_2^T y^T + y D_4 y^T \\
&\geq y(D_4 - D_2 D_1^{-1} D_2^T) y^T.
\end{aligned}$$

Then

$$\begin{aligned}
\int_{\mathbb{R}^{(1+\#I_{oe}^\sigma)d}} \sup_x p_n(\frac{x}{n^H}, y)\, dy &= \int_{\mathbb{R}^{(1+\#I_{oe}^\sigma)d}} \sup_x p_n(x,y)\, dy \\
&= c_4 \int_{\mathbb{R}^{(1+\#I_{oe}^\sigma)d}} (\det(Q_n))^{-\frac{1}{2}} \sup_x \exp\Big\{ -\frac{1}{2}(x,y) Q_n^{-1} (x,y)^T \Big\}\, dy \\
&\leq c_4 \int_{\mathbb{R}^{(1+\#I_{oe}^\sigma)d}} (\det(Q_n))^{-\frac{1}{2}} \exp\Big\{ -\frac{1}{2} y\big(D_4 - D_2 D_1^{-1} D_2^T\big) y^T \Big\}\, dy \\
&= c_5 (\det(Q_n))^{-\frac{1}{2}} \big(\det(D_4 - D_2 D_1^{-1} D_2^T)\big)^{\frac{1}{2}} \\
&= c_5 (\det(A))^{-\frac{1}{2}}.
\end{aligned}$$

Therefore,

$$n^{m(2+Hd)} \Big| \mathbb{E}\Big[\int_{D^n_{K,\epsilon}} \prod_{j=1}^{2m} f\big(n^H B^{H,1}_{u_j} - n^H B^{H,2}_{v_{\sigma(j)}}\big)\, du\, dv \Big] \Big| \leq c_6\, n^{-Hd} \int_{\widehat{D}^n_{K,\epsilon}} (\det(A(w,s)))^{-\frac{1}{2}}\, dw\, ds.$$

From the definition of X in (3.18), we see that the components of X are linearly independent and thus $A(w,s)$ is not singular. Taking into account the definition of $\widehat{D}^n_{K,\epsilon}$ in (3.17) and the continuity of $\det(A(w,s))$, we obtain $\det(A(w,s)) \geq c_\epsilon > 0$ for all (w,s) in $\widehat{D}^n_{K,\epsilon}$. Therefore,

$$n^{m(2+Hd)} \Big| \mathbb{E}\Big[\int_{D^n_{K,\epsilon}} \prod_{j=1}^{2m} f\big(n^H B^{H,1}_{u_j} - n^H B^{H,2}_{v_{\sigma(j)}}\big)\, du\, dv \Big] \Big| \leq c_7\, c_\epsilon^{-\frac{1}{2}} n^{-Hd}.$$

This completes the proof. ■

We next show the convergence of all even moments.

Proposition 3.9 *For any $m \in \mathbb{N}$,*

$$\lim_{n\to\infty} \mathbb{E}\left[F_n(t_1,t_2)\right]^{2m} = D_{H,d}^m \, \|f\|^m_{\frac{2}{H}-d} \, \mathbb{E}\left[\sqrt{\alpha(0,t_1,t_2)}\,\zeta\right]^{2m}.$$

Proof. By Proposition 3.8 and equation (3.11), we only need to show

$$\begin{aligned}\lim_{n\to\infty} (2m)!\, n^{m(2+Hd)} \sum_{\sigma\in\mathscr{P}_0} \mathbb{E}\Big[\int_D \prod_{j=1}^{2m} f\big(n^H B^{H,1}_{u_j} - n^H B^{H,2}_{v_{\sigma(j)}}\big)\, du\, dv\Big] \\ = D_{H,d}^m \, \|f\|^m_{\frac{2}{H}-d} \, \mathbb{E}\left[\sqrt{\alpha(0,t_1,t_2)}\,\zeta\right]^{2m}. \qquad (3.19)\end{aligned}$$

The proof will be done in several steps.

Step 1 Since $\sigma \in \mathscr{P}_0$, by Lemma 3.7, the equation (3.15) can be written as

$$\begin{aligned}&n^{2m}\mathbb{E}\Big[\int_D \prod_{j=1}^{2m} f\big(n^H B^{H,1}_{u_j} - n^H B^{H,2}_{v_{\sigma(j)}}\big)\, du\, dv\Big] \\ &= \mathbb{E}\Big[\int_{D_n} \prod_{j\in I^\sigma_{ee}} f\big(n^H[B^{H,1}_{w_j} - B^{H,2}_{s_{\sigma(j)}}]\big) \prod_{j\in I^\sigma_{eo}} f\big(n^H[B^{H,1}_{w_j} - B^{H,2}_{s_{\sigma(j)+1}}] - n^H[B^{H,2}_{s_{\sigma(j)+1}-\frac{s_{\sigma(j)}}{n}} - B^{H,2}_{s_{\sigma(j)+1}}]\big) \\ &\quad\times \prod_{j\in I^\sigma_{eo}} f\big(n^H[B^{H,1}_{w_j} - B^{H,2}_{s_{\sigma(j)+1}}] + n^H[B^{H,1}_{w_j-\frac{w_{j-1}}{n}} - B^{H,1}_{w_j}]\big) \\ &\qquad\times \prod_{j\in I^\sigma_{ee}} f\big(n^H[B^{H,1}_{w_j} - B^{H,2}_{s_{\sigma(j)}}] + n^H[B^{H,1}_{w_j-\frac{w_{j-1}}{n}} - B^{H,1}_{w_j}] - n^H[B^{H,2}_{s_{\sigma(j)}-\frac{s_{\sigma(j)-1}}{n}} - B^{H,2}_{s_{\sigma(j)}}]\big)\, dw\, ds\Big].\end{aligned}$$

We introduce random fields $X^n(w,s) = \big\{X^n_j(w,s) : j \in I_e\big\}$ and $Y^n(w,s) = \big\{Y^n_j(w,s) : j \in I_e\big\}$ with

$$X^n_j(w,s) = \begin{cases} B^{H,1}_{w_j} - B^{H,2}_{s_{\sigma(j)}}, & \text{if } j \in I^\sigma_{ee}, \\ [B^{H,1}_{w_j} - B^{H,2}_{s_{\sigma(j)+1}}] - [B^{H,2}_{s_{\sigma(j)+1}-\frac{s_{\sigma(j)}}{n}} - B^{H,2}_{s_{\sigma(j)+1}}], & \text{if } j \in I^\sigma_{eo}, \end{cases}$$

and

$$Y^n_j(w,s) = \begin{cases} n^H[B^{H,1}_{w_j-\frac{w_{j-1}}{n}} - B^{H,1}_{w_j}] - n^H[B^{H,2}_{s_{\sigma(j)}-\frac{s_{\sigma(j)-1}}{n}} - B^{H,2}_{s_{\sigma(j)}}], & \text{if } j \in I^\sigma_{ee}, \\ n^H[B^{H,1}_{w_j-\frac{w_{j-1}}{n}} - B^{H,1}_{w_j}] + n^H[B^{H,2}_{s_{\sigma(j)+1}-\frac{s_{\sigma(j)}}{n}} - B^{H,2}_{s_{\sigma(j)+1}}], & \text{if } j \in I^\sigma_{eo}. \end{cases}$$

Let $Z_n(w,s) = \big(X^n(w,s), Y^n(w,s)\big)$. Denote the covariance matrix and the probability density function of the Gaussian random field $Z_n(w,s)$ by $Q_n(w,s)$ and

$$p_n(x,y) = (2\pi)^{-md}\big(\det Q_n(w,s)\big)^{-\frac{1}{2}} \exp\Big\{-\frac{1}{2}(x,y)Q_n(w,s)^{-1}(x,y)^T\Big\},$$

respectively. Then

$$
\begin{aligned}
&n^{m(2+Hd)}\mathbb{E}\Big[\int_D \prod_{j=1}^{2m} f\big(n^H B^{H,1}_{u_j} - n^H B^{H,2}_{v_{\sigma(j)}}\big)\,du\,dv\Big] \\
&= n^{mHd}\mathbb{E}\Big[\int_{D_n} \prod_{j\in I_e} f\big(n^H X^n_j(w,s)\big) f\big(n^H X^n_j(w,s) + Y^n_j(w,s)\big)\,dw\,ds\Big] \\
&= n^{mHd}\int_{\mathbb{R}^{2md}}\int_{D_n} \prod_{j\in I_e} f(n^H x_j) f(n^H x_j + y_{j-1})\,p_n(x,y)\,dw\,ds\,dx\,dy \\
&= \int_{\mathbb{R}^{2md}}\int_{D_n} F(x,y)\,p_n(\frac{x}{n^H},y)\,dw\,ds\,dx\,dy, \qquad (3.20)
\end{aligned}
$$

where $F(x,y) = \prod_{j\in I_e} f(x_j)f(x_j+y_j)$.

We need to compute the limit of the density $p_n(x/n^H, y)$ as n tends to infinity. The covariance matrix between the components of $X^n(w,s)$ and $Y^n(w,s)$ converges to the zero matrix, and the covariance matrix of the random field $Y^n(w,s)$ converges to a diagonal matrix with entries equal to $w^{2H}_{j-1} + s^{2H}_{\sigma(j)-1}$ when $j \in I^\sigma_{ee}$ and $w^{2H}_{j-1} + s^{2H}_{\sigma(j)}$ when $j \in I^\sigma_{eo}$. Let $A^\sigma(w,s)$ be the covariance matrix of $X(w,s) = (X_j(w,s) : j \in I_e)$ with

$$
X_j(w,s) = \begin{cases} B^{H,1}_{w_j} - B^{H,2}_{s_{\sigma(j)}}, & \text{if } j \in I^\sigma_{ee}, \\ \\ B^{H,1}_{w_j} - B^{H,2}_{s_{\sigma(j)+1}}, & \text{if } j \in I^\sigma_{eo}. \end{cases}
$$

We see that the covariance matrix of the random field $X^n(w,s)$ converges to $A^\sigma(w,s)$. Thus,

$$
\begin{aligned}
\lim_{n\to\infty} p_n(\frac{x}{n^H},y) = (2\pi)^{-md}\big(\det A^\sigma(w,s)\big)^{-\frac12} \\
\times \prod_{j\in I_{ee}} \big(w^{2H}_{j-1} + s^{2H}_{\sigma(j)-1}\big)^{-\frac{d}{2}} \exp\Big(-\frac12 \frac{|y_j|^2}{w^{2H}_{j-1} + s^{2H}_{\sigma(j)-1}}\Big) \\
\times \prod_{j\in I_{eo}} \big(w^{2H}_{j-1} + s^{2H}_{\sigma(j)}\big)^{-\frac{d}{2}} \exp\Big(-\frac12 \frac{|y_j|^2}{w^{2H}_{j-1} + s^{2H}_{\sigma(j)}}\Big).
\end{aligned}
$$

On the other hand, the region D_n converges, as n tends to infinity, to

$$
\Big\{w,s \in \mathbb{R}^{2m}_+ : 0 < w_2 < w_4 < \cdots < w_{2m} < t_1,\ 0 < s_2 < s_4 < \cdots < s_{2m} < t_2\Big\}.
$$

Note that we can add a term -1 because $\int_{\mathbb{R}^d} F(x,y)\,dy_j = 0$ for all j in I_e and

$$
\begin{aligned}
&\int_0^\infty\int_0^\infty (w^{2H}+s^{2H})^{-\frac{d}{2}}\big(e^{-\frac12\frac{|y_j|^2}{w^{2H}+s^{2H}}} - 1\big)\,dw\,ds \\
&= -|y_j|^{\frac{2}{H}-d}\int_0^\infty\int_0^\infty (w^{2H}+s^{2H})^{-\frac{d}{2}}\big(1 - e^{-\frac12\frac{1}{w^{2H}+s^{2H}}}\big)\,dw\,ds.
\end{aligned}
$$

Therefore, provided that we can interchange the limit and the integrals in the expression (3.20), we obtain that the limit equals

$$
\frac{D^m_{H,d}}{4^m}\|f\|^m_{\frac{2}{H}-d}(2\pi)^{-\frac{md}{2}}\int_O \big(\det A^\sigma(w,s)\big)^{-\frac12} dw\,ds, \qquad (3.21)
$$

where

$$O = \Big\{0 < w_2 < w_4 < \cdots < w_{2m} < t_1,\, 0 < s_2 < s_4 < \cdots < s_{2m} < t_2\Big\}.$$

Finally, the left hand side of (3.19) equals

$$\begin{aligned}
&(2m)! \sum_{\sigma \in \mathscr{P}_0} \frac{D_{H,d}^m}{4^m} \|f\|^m_{\frac{2}{H}-d} (2\pi)^{-md} \int_O (\det A^\sigma(w,s))^{-\frac{1}{2}} dw\, ds \\
&= (2m-1)!! D_{H,d}^m \|f\|^m_{\frac{2}{H}-d} \int_{E^m} (2\pi)^{-\frac{md}{2}} (\det A(u,v))^{-\frac{1}{2}} du\, dv,
\end{aligned}$$

and, taking into account of Lemma 2.2, this would finish the proof.

Step 2 Recall the notation D_n in (3.14). Define

$$D_{n,K} = D_n \cap \Big\{0 < w_{2k-1} < K \wedge n(w_{2k} - w_{2k-2}),\, 0 < s_{2k-1} < K \wedge n(s_{2k} - s_{2k-2}),\, 1 \le k \le m\Big\}.$$

The region $D_{n,K}$ is uniformly bounded in n and we can then interchange the limit and the integral with respect to w and s, provided that we have a uniform integrability condition.

Observe that

$$\begin{aligned}
&\int_{D_{n,K}} \big|p_n(\frac{x}{n^H}, y)\big|^p \, dw\, ds \\
&\le c_1 \int_{D_{n,K}} \big(\det Q_n(w,s)\big)^{-\frac{p}{2}} \, ds\, ds \\
&= c_2 \int_{D_{n,K}} \Big(\int_{\mathbb{R}^{2md}} \exp\Big\{ -\frac{1}{2} \mathrm{Var}\Big(\sum_{j\in I_e} \xi_j \cdot X_j^n(w,s) + \sum_{j \in I_e} \eta_j \cdot Y_j^n(w,s) \Big) \Big\} d\xi\, d\eta \Big)^p \, dw\, ds
\end{aligned}$$

and

$$\mathrm{Var}\Big(\sum_{j\in I_e} \xi_j \cdot X_j^n(w,s) + \sum_{j \in I_e} \eta_j \cdot Y_j^n(w,s) \Big) = I_1(\xi,\eta) + I_2(\xi,\eta),$$

where

$$I_1(\xi,\eta) = \mathrm{Var}\Big(\sum_{j\in I_e} \xi_j \cdot B^{H,1}_{w_j} + \sum_{j\in I_e} \eta_j \cdot n^H [B^{H,1}_{w_j - \frac{w_{j-1}}{n}} - B^{H,1}_{w_j}] \Big)$$

and

$$\begin{aligned}
I_2(\xi,\eta) = \mathrm{Var}\Big(&\sum_{j \in I^\sigma_{ee}} \xi_j \cdot B^{H,2}_{s_{\sigma(j)}} + \sum_{j\in I^\sigma_{eo}} \xi_j \cdot (B^{H,2}_{s_{\sigma(j)+1}} + [B^{H,2}_{s_{\sigma(j)+1} - \frac{s_{\sigma(j)}}{n}} - B^{H,2}_{s_{\sigma(j)+1}}]) \\
&+ \sum_{j\in I^\sigma_{ee}} \eta_j \cdot n^H [B^{H,2}_{s_{\sigma(j)} - \frac{s_{\sigma(j)-1}}{n}} - B^{H,2}_{s_{\sigma(j)}}] + \sum_{j \in I^\sigma_{eo}} \eta_j \cdot n^H [B^{H,2}_{s_{\sigma(j)+1} - \frac{s_{\sigma(j)}}{n}} - B^{H,2}_{s_{\sigma(j)+1}}] \Big).
\end{aligned}$$

Applying Cauchy-Schwartz inequality gives

$$\begin{aligned}
&\int_{\mathbb{R}^{2md}} \exp\Big\{ -\frac{1}{2}\big[I_1(\xi,\eta) + I_2(\xi,\eta)\big]\Big\} \, d\xi\, d\eta \\
&\le \Big(\int_{\mathbb{R}^{2md}} \exp\big\{ -I_1(\xi,\eta)\big\} \, d\xi\, d\eta \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^{2md}} \exp\big\{ -I_2(\xi,\eta)\big\} \, d\xi\, d\eta \Big)^{\frac{1}{2}}.
\end{aligned}$$

Using similar arguments as in the proof of Proposition 3.4 in [3], we obtain

$$\int_{\mathbb{R}^{2md}} \exp\big\{ -I_1(\xi,\eta)\big\} \, d\xi\, d\eta \le c_3 \prod_{k=1}^m (u_{2k-1})^{-Hd} \big(u_{2k} - \frac{u_{2k-1}}{n} - u_{2k-2}\big)^{-Hd}$$

and

$$\int_{\mathbb{R}^{2md}} \exp\big\{-I_2(\xi,\eta)\big\}\, d\xi\, d\eta \le c_4 \prod_{k=1}^{m} (v_{2k-1})^{-Hd}\big(v_{2k} - \frac{v_{2k-1}}{n} - v_{2k-2}\big)^{-Hd}.$$

Therefore, for all p such that $1 \le p < \frac{2}{Hd}$,

$$\sup_n \int_{D_{n,K}} \big|p_n(\frac{x}{n^H}, y)\big|^p \, dw\, ds \le c_5,$$

where c_5 is a positive constant independent of n, x and y.

Let

$$I_{n,K} = \int_{\mathbb{R}^{2md}} \int_{D_{n,K}} F(x,y)\, p_n(\frac{x}{n^H}, y)\, dw\, ds\, dx\, dy.$$

Thus, taking into account that the function $F(x,y)$ is continuous and has compact support, by the dominated convergence theorem, we obtain

$$\lim_{n\to\infty} I_{n,K} = \int_{\mathbb{R}^{2md}} F(x,y) \Big(\lim_{n\to\infty} \int_{D_{n,K}} p_n(\frac{x}{n^H}, y)\, dw\, ds \Big)\, dx\, dy.$$

On the other hand, there exists $p > 1$ such that

$$\sup_n \int_{D_{n,K}} |p_n(\frac{x}{n^H}, y)|^p \, dw\, ds < \infty,$$

which implies

$$\lim_{n\to\infty} I_{n,K} = \int_{\mathbb{R}^{2md}} \int_{\mathbb{R}^{2m}} F(x,y) \lim_{n\to\infty} \mathbf{1}_{D_{n,K}}((w,s))\, p_n(\frac{x}{n^H}, y)\, dw\, ds\, dx\, dy.$$

With the same notation as in **Step 1** we get

$$\begin{aligned}\lim_{n\to\infty} I_{n,K} = &\frac{(2m)!}{(2\pi)^{md}} \Big(\int_O \big[\det A^\sigma(w,s)\big]^{-\frac{1}{2}} dw\, ds \Big) \\ &\times \int_{\mathbb{R}^{2md}} F(x,y) \prod_{j\in I_e} \int_0^K \int_0^K (u^{2H} + v^{2H})^{-\frac{d}{2}} \big(e^{-\frac{1}{2}\frac{|y_j|^2}{u^{2H}+v^{2H}}} - 1\big)\, du\, dv\, dx\, dy.\end{aligned}$$

The right hand side of the above equality converges to the term in (3.21) as K tends to infinity.

Step 3 We need to show that

$$\lim_{K\to\infty} \limsup_{n\to\infty} \int_{\mathbb{R}^{2md}} \int_{D_n - D_{n,K}} F(x,y)\, p_n(\frac{x}{n^H}, y)\, dw\, ds\, dx\, dy = 0. \tag{3.22}$$

Recall the equation (3.20) and the notation $D^n_{K,\ell}$ in (3.13).

$$\begin{aligned}&\int_{\mathbb{R}^{2md}} \int_{D_n - D_{n,K}} F(x,y)\, p_n(\frac{x}{n^H}, y)\, dw\, ds\, dx\, dy \\ &\qquad = n^{m(2+Hd)} \mathbb{E}\Big[\int_{\cup_{\ell=1}^m D^n_{K,\ell}} \prod_{j=1}^{2m} f\big(n^H B^{H,1}_{u_j} - n^H B^{H,2}_{v_{\sigma(j)}}\big)\, du\, dv \Big].\end{aligned}$$

Therefore, the statement in (3.22) follows from Lemma 3.6. The proof is completed. ∎

Proof of Theorem 1.1. This follows from Propositions 3.1, 3.3 and 3.9 by the method of moments.

4 Appendix

Here we give some lemmas which are necessary in the proof of Theorem 1.1.

Lemma 4.1 *Assume that $1 < Hd < 2$. There exists a positive contant c such that*

$$\int_0^a \int_0^b (w^{2H} + s^{2H})^{-\frac{d}{2}} \, dw \, ds \le c\,(a \wedge b)^{2-Hd}.$$

Proof. Without loss of generality, we can assume that $a \le b$. Making the change of variable $v = s/w$ gives

$$\begin{aligned}
\int_0^a \int_0^b (w^{2H} + s^{2H})^{-\frac{d}{2}} \, dw \, ds &= \int_0^a \int_0^{\frac{b}{w}} w^{1-Hd}(1+v^{2H})^{-\frac{d}{2}} \, dv \, dw \\
&\le \int_0^a \int_0^\infty w^{1-Hd}(1+v^{2H})^{-\frac{d}{2}} \, dv \, dw \\
&\le c_1 a^{2-Hd},
\end{aligned}$$

where c_1 is a positive constant independent of b. ■

Lemma 4.2 *Assume that $2 - H < Hd < 2$. Let X be a d-dimensional centered normal random vector with covariance matrix $\sigma^2 I$. Then, for any $n \in \mathbb{N}$ and $y \in \mathbb{R}^d$, there exists a positive constant c depending only on H and d such that*

$$\int_0^\infty \int_0^\infty (w^{2H} + s^{2H})^{-\frac{d}{2}} \, \mathbb{E} \Big| \exp\big(\iota \frac{y \cdot X}{n^H \sqrt{w^{2H} + s^{2H}}}\big) - 1 \Big| \, dw \, ds \le c\, n^{Hd-2} |y|^{\frac{2}{H} - d}.$$

Proof. It suffices to show the above inequality when $y \neq 0$. Making the change of variables $u = |y|^{-\frac{1}{H}} nw$ and $v = |y|^{-\frac{1}{H}} ns$ gives

$$\begin{aligned}
&\int_0^\infty \int_0^\infty (w^{2H} + s^{2H})^{-\frac{d}{2}} \, \mathbb{E} \Big| \exp\big(\iota \frac{y \cdot X}{n^H \sqrt{w^{2H} + s^{2H}}}\big) - 1 \Big| \, dw \, ds \\
&= n^{Hd-2} |y|^{\frac{2}{H} - d} \int_0^\infty \int_0^\infty (u^{2H} + v^{2H})^{-\frac{d}{2}} \, \mathbb{E} \Big| \exp\big(\iota \frac{y \cdot X}{|y| \sqrt{u^{2H} + v^{2H}}}\big) - 1 \Big| \, du \, dv \\
&\le n^{Hd-2} |y|^{\frac{2}{H} - d} \int_0^\infty \int_0^\infty (u^{2H} + v^{2H})^{-\frac{d}{2}} \Big(2 \wedge (u^{2H} + v^{2H})^{-\frac{1}{2}} \mathbb{E}\, |X|\Big) \, du \, dv \\
&= c\, n^{Hd-2} |y|^{\frac{2}{H} - d}.
\end{aligned}$$

The last equality follows from using polar coordinates and the assumption $2 - H < Hd < 2$. ■

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